

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Discrete Mathematics 290 (2005) 221–228

DISCRETE
MATHEMATICSwww.elsevier.com/locate/disc

Antipodality properties of finite sets in Euclidean space

H. Martini^a, V. Soltan^{b, 1}^a*Fakultät für Mathematik TU Chemnitz, D-09107 Chemnitz, Germany*^b*Department of Mathematical Sciences, George Mason University, 4400 University Drive, Fairfax, VA 22030-4444, USA*

Received 14 August 2003; received in revised form 2 September 2004; accepted 21 September 2004
Available online 11 January 2005

Abstract

This is a survey of known results and still open problems on antipodal properties of finite sets in Euclidean space. The exposition follows historical lines and takes into consideration both metric and affine aspects.

© 2004 Elsevier B.V. All rights reserved.

Keywords: Affine diameter; Metric diameter; Double normal; Antipodal points; Finite set; Convex polytope

1. Introduction

Although some antipodality properties of finite sets in Euclidean space were known for a long period, they became a focus of proper interest in the second half of the 20th century, especially with the development of discrete and combinatorial geometry.

This article gives a survey of known combinatorial results on antipodal points of finite sets in Euclidean space E^d . The exposition follows historical lines and takes into consideration both metric and affine aspects, including metric antipodals, double normals, and affine diameters.

Various results on antipodality properties are covered by the surveys Erdős and Purdy [16], Grünbaum [19], and Martini [29], as well as by the monographs Boltyanski and Gohberg

¹ Supported by Deutscher Akademischer Austauschdienst.

E-mail address: vsoltan@gmu.edu (V. Soltan).

[3], Boltyanski et al. [4], Pach and Agarwal [33]. This article gives a complete picture of the field under consideration, with an explicit description of still open problems.

Throughout the paper, we will be dealing with subsets of the d -dimensional Euclidean space E^d , and $\|\cdot\|$ will denote the norm in E^d . In what follows, X means a finite set in E^d , not lying in a hyperplane, and X_n stands for a finite set of n points in E^d .

Regarding the antipodality properties of a set $X \subset E^d$, we will consider pairs of points from X that satisfy a given property \mathcal{F} (which is a binary relation on X), like to be metric antipodals, to form a double normal, etc. With respect to this property, the following two types of problems will be studied:

1. How many points can a finite set $X \subset E^d$ have, such that any pair of them has property \mathcal{F} ?
2. How many pairs of points with property \mathcal{F} can a set $X_n \subset E^d$ of n points have?

2. Metric antipodals

Obviously, any finite set $X \subset E^d$ contains a pair of points a, b with the property

$$\|a - b\| = \max\{\|x - y\| : x, y \in X\}.$$

The distance $\|a - b\|$ is called the (metric) *diameter* of X , and points a, b are called *metric antipodals*. If H_a and H_b are the hyperplanes orthogonal to the line segment $[a, b]$ such that $a \in H_a$ and $b \in H_b$, then, as easily seen, X lies entirely in the slab between H_a and H_b (see, e.g., [5, p. 51]). Moreover, $X \cap H_a = \{a\}$ and $X \cap H_b = \{b\}$.

Denote by $M(d)$ the maximum cardinality of a finite set $X \subset E^d$ with the property that any two points of X are metric antipodals. Also, let $M_d(X_n)$ denote the number of pairs of metrically antipodal points in a set $X_n \subset E^d$, and let $M_d(n)$ be the maximum of $M_d(X_n)$ taken over all sets X_n of n elements in E^d .

The problem on the value for $M(d)$ is rather simple: $M(d) = d + 1$, and a set $X_{d+1} \subset E^d$ such that any two of its points are metric antipodals is the vertex set of a regular d -simplex.

A similar problem on the value for $M_d(n)$ was posed by Erdős [10] in 1946 and solved by him for the case $d = 2$: $M_2(n) = n$. Erdős mistakenly attributed this problem to Hopf and Pannwitz [25], who posed in 1934 the following particular question related to the topic in discussion: Is it true that for any n (≥ 5) distinct points $a_0, a_1, \dots, a_{n-1}, a_n = a_0$ in the plane, the relations

$$\|a_i - a_j\| \leq 1, \quad \|a_i - a_{i+1}\| = 1, \quad i = 0, 1, \dots, n-1,$$

are possible if and only if n is odd? Positive answers, by Fenchel [17] and Sutherland [36], were published a year later, followed by the editors' remark that similar solutions were submitted by many others.

Neaderhouser and Purdy [30], based on Woodall's study of the structure of certain finite graphs in the plane (see [39]), gave a complete characterization of those sets $X_n \subset E^2$ that have exactly n pairs of metric antipodals; they proved that a set $X_n \subset E^2$ satisfies the condition $M_2(X_n) = n$ if and only if it can be represented as $X = Y \cup Z$, where

$Y = \{y_1, \dots, y_m\}$ is the vertex set of a Reuleaux m -gon B and $Z = \{z_1, \dots, z_{n-m}\}$ is a point set disjoint with Y and lying in the boundary of B .

In the same paper [10], Erdős asked for an analogous result for n points in higher dimension and cited Vázsonyi's conjecture (oral communication) that in three-dimensional space the maximum distance in any set X_n cannot occur more than $2n - 2$ times, i.e., that $M_3(n) \leq 2n - 2$. This conjecture was independently proved by Grünbaum [18], Heppes [24], and Straszewicz [35], all of them using a similar technique. Each of these three authors gave an example of a set $X_n \subset E^3$, with $n \geq 4$, which has exactly $2n - 2$ pairs of metric antipodals, thus showing that $M_3(n) = 2n - 2$. In an attempt to describe all sets $X_n \subset E^3$ with exactly $2n - 2$ pairs of metric antipodals, Neaderhouser and Purdy [30] found three series of such sets and asked whether there are other types of sets with this property.

Regarding the same problem in dimensions $d \geq 4$, Hadwiger [22] conjectured that $M_d(n) < n(d + 1)/2$. This conjecture was disproved by Lenz (see an announcement in [23]), who constructed a set $X_n \subset E^d$ with at least $(d - 1)n - (d + 1)(d - 2)/2$ pairs of metric antipodals. Furthermore, Lenz constructed another example (see [13] for details) of a set $X_n \subset E^d$ with at least $\lfloor n^2/4 \rfloor$ pairs of metric antipodals, implying that $M_d(n) \geq \lfloor n^2/4 \rfloor$. As mentioned by Pach and Agarwal [33], a slight modification of Lenz's example gives an improved inequality $M_d(n) > n^2/4 + cn$ for a certain $c > 0$. Answering Lenz's question on the value for the limit of $M_d(n)/n^2$ as $n \rightarrow \infty$, Erdős [13] proved that, for every $d \geq 4$,

$$\lim_{n \rightarrow \infty} \frac{M_d(n)}{n^2} = \frac{1}{2} - \frac{1}{2\lfloor d/2 \rfloor}.$$

From the results of Erdős and Pach [15] (see also [33]) it also follows that

$$M_d(n) \geq \frac{n^2}{2} \left(1 - \frac{1}{\lfloor d/2 \rfloor} + o(1) \right)$$

provided $d \geq 4$ and $n \rightarrow \infty$.

Asking for exact values for $M_d(n)$ with small integers $n \geq d + 1$, Yugaï [40] proved the equalities

$$M_d(n) = \binom{d+1}{2} + (d-1)(n-d-1), \quad d \geq 2, \quad d+1 \leq n \leq d+3.$$

In particular, $M_4(6) = 13$ and $M_4(7) = 16$.

Problem 1. What are the exact values for $M_d(n)$, $d \geq 4$, $n \geq d + 1$, and what is the description of the sets $X_n \subset E^d$ with the property $M_d(X_n) = M_d(n)$?

3. Double normals

A pair of points a, b in a set $X \subset E^d$ is called a *double normal* provided X lies in the slab between the hyperplanes H_a and H_b through a and b , respectively, both orthogonal to $[a, b]$. The double normal $\{a, b\}$ is called *strict* provided $X \cap H_a = \{a\}$ and $X \cap H_b = \{b\}$. Clearly, $\{a, b\}$ is a double normal (strict double normal) if and only if, for any point $c \in X \setminus \{a, b\}$, the triangle $\Delta(a, b, c)$ is not obtuse (respectively, is acute). Obviously, a pair $\{a, b\}$ of metric

antipodals in a set X is a strict double normal of X . We remark here that the notion of double normal is useful in convex geometry, especially in the study of bodies of constant width (see, e.g., [6]).

Denote by $N(d)$ (respectively, by $N'(d)$) the maximum cardinality of a finite set $X \subset E^d$ with the property that any two points of X form a double normal (respectively, a strict double normal). Also, let $N_d(X_n)$ (respectively, $N'_d(X_n)$) denote the number of double normals (respectively, strict double normals) of a set $X_n \subset E^d$.

In 1957, Erdős [12] conjectured that a set $X \subset E^d$ such that any two of its points form a double normal having at most 2^d points. Erdős mentioned that the case $d = 2$ is trivial, and that an unpublished proof has been given by Kuiper and Boerdijk for $d = 3$. Croft [7] stated (without proof) that a set of seven or eight points in E^3 such that any two of them form a double normal belong to the vertex set of a rectangular parallelepiped.

Erdős' conjecture above was positively solved in 1962 by Danzer and Grünbaum [9] in a more general setting (see Section 4 below). From their proof it follows that a set $X \subset E^d$ of 2^d points with the property that any two of its points form a double normal is the vertex set of a rectangular parallelotope. As a consequence, $N(d) = 2^d$.

In 1948, Erdős [11] modified the problem above, asking whether any set of eight points in E^3 contains three points that do not form an acute triangle. In other words, Erdős asked whether $N'(3) < 8$. In 1957, Erdős [12] sharpened this problem by lowering the expected number of points to six. A positive answer to the last version of the problem was given by Croft [7], who showed that $N'(3) = 5$. Since Croft's proof was rather long, Schütte [34] gave a shorter one.

Regarding a similar question in higher dimensions, Danzer and Grünbaum [9] (see also [20,21]) gave an example of a set X of $2d - 1$ points in E^d such that any two of them form a strict double normal. It was believed for a long time that $N'(d) = 2d - 1$, but Erdős and Füredi [14] (see also [1]) showed the existence of a set X of cardinality $\left\lfloor \left(2/\sqrt{3}\right)^d / 2 \right\rfloor \approx (1.15)^d$ in E^d such that any two of its points form a strict double normal, and without proof they announced that this lower bound for X can be replaced by $\left(\sqrt[4]{2} - o(1)\right)^{d-1}$, which is about $(1.19)^d$. Erdős and Füredi [14] also conjectured the existence of an absolute constant $c > 0$ (not depending on d) such that $N'(d) \leq (2 - c)^d$.

Problem 2. What are the exact values for $N'(d)$, $d \geq 4$?

It is interesting to observe the absence of any estimates for the numbers $N_d(X_n)$ and $N'_d(X_n)$. This, probably, happened due to the study of the affine variant of these problems (see Section 4).

Problem 3. What are the exact values for $N_d(n)$, $d \geq 2$, $n \geq d + 1$, and what is the description of the sets $X_n \subset E^d$ with the property $N_d(X_n) = N_d(n)$?

Problem 4. What are the exact values for $N'_d(n)$, $d \geq 2$, $n \geq d + 1$, and what is the description of the sets $X_n \subset E^d$ with the property $N'_d(X_n) = N'_d(n)$?

4. Affine antipodals

A pair of points a, b in a set $X \subset E^d$ is called (affinely) *antipodal* provided there are distinct parallel hyperplanes H_a and H_b through a and b , respectively, such that X lies in the slab between H_a and H_b . Moreover, the pair $\{a, b\}$ is called *strictly antipodal* provided $X \cap H_a = \{a\}$ and $X \cap H_b = \{b\}$. Obviously, any (strict) double normal of X is a (strict) antipodal pair of X . Note that antipodal pairs of points in various sets play a role in Minkowski geometry (cf. [38, Section 4.8]).

Denote by $A(d)$ the maximum cardinality of a finite set $X \subset E^d$ with the property that any two points of X are antipodal. Also, let $A_d(X_n)$ denote the number of antipodal pairs in a set $X_n \subset E^d$, and let $A_d(n)$ be the maximum of $A_d(X_n)$ taken over all sets X_n with n elements in E^d . A similar notation, $A'(d)$, $A'_d(X_n)$, $A'_d(n)$, is used when strictly antipodal pairs are considered instead of antipodal pairs.

In 1960, Klee [26] conjectured that a set $X \subset E^d$ such that any two points of X are antipodal has at most 2^d points. Klee's conjecture was positively solved in 1962 by Danzer and Grünbaum [9]. In particular, they showed that a set $X \subset E^d$ of 2^d points such that any two of its points are antipodal is the vertex set of a d -parallelotope. Thus $A(d) = 2^d$.

A variation of Klee's problem was recently proposed by Talata [37]. A convex polytope $P \subset E^d$ is called *edge-antipodal* provided the endpoints of any edge $[a, b]$ of P are antipodal. Talata asked about the existence of a positive integer m such that any edge-antipodal polytope $P \subset E^3$ has at most m vertices. Csikós [8] showed that any edge-antipodal polytope in E^3 has at most 12 vertices, asking for a better upper bound. Finally, Bezdek et al. [2] proved that an edge-antipodal 3-polytope in E^3 has at most 8 vertices, with equality only for affine cubes. As shown by Talata [8], there are edge-antipodal polytopes in E^d , $d \geq 4$, having pairs of non-antipodal vertices.

In 1963, Grünbaum [20] posed the problem on the upper bound for $A_d(V_n)$, where V_n is the vertex set of a convex d -polytope, and stated that for $d = 2$ the respective maximum equals $\lfloor 3n/2 \rfloor$. For the case $d = 3$, he announced (see [21]) that

$$A_3(V_n) \geq \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{(n+1)}{2} \right\rfloor + \left\lfloor \frac{3n}{2} \right\rfloor + \left\lfloor \frac{(3n+1)}{4} \right\rfloor.$$

Nguyen and Soltan [32] sharpened these results by showing that $A_2(V_n) = n + k$, where k is the number of pairs of parallel sides of the polygon, which is obtained as convex hull of V_n , and that, for all $d \geq 3$,

$$A_d(V_n) \geq n - 1 + \frac{d(d-1)}{2},$$

with equality if and only if V_n contains $d + 1$ pairwise antipodal points and any other point from V_n is antipodal to exactly one of these $d + 1$ points.

If $V_d(n)$ denotes the maximum of $A_d(V_n)$ over all vertex sets V_n of convex d -polytopes in E^d , then Grünbaum [21] asked whether the relation

$$\lim_{n \rightarrow \infty} \frac{V_d(n)}{n^2} = \frac{1}{2} - \frac{1}{2^{d-1}}$$

holds for all $d \geq 2$.

Makai and Martini [27] proved that

$$\left\lfloor \frac{n^2}{2} \right\rfloor \leq A_2(n) \leq \frac{n^2}{4} + O(n), \quad \left\lfloor \frac{3n^2}{8} + 4 \right\rfloor \leq A_3(n) \leq \frac{7n^2}{16},$$

and, in particular,

$$\left\lfloor \frac{n^2}{3} \right\rfloor \leq V_3(n) \leq \frac{7n^2}{16}.$$

The last inequality disproves Grünbaum's conjecture on the value for the limit of $V_d(n)/n^2$ as $n \rightarrow \infty$.

Moreover, Makai and Martini [27] proved that, for all $d \geq 4$,

$$\left(1 - \frac{1}{3 \cdot 2^{d-3}}\right) \frac{n^2}{2} - O(1) \leq V_d(n) \leq \left(1 - \frac{1}{2^d}\right) \frac{n^2}{2}$$

and

$$\left(1 - \frac{1}{2^{d-1}}\right) \frac{n^2}{2} - O(1) \leq A_d(n) \leq \left(1 - \frac{1}{2^d}\right) \frac{n^2}{2}.$$

They also conjectured that, for all $d \geq 2$,

$$A_d(n) \leq \left(1 - \frac{1}{2^{d-1}}\right) \frac{n^2}{2}.$$

Problem 5. What are the exact values for $A_d(n)$, $d \geq 3$, $n \geq d+1$, and what is the description of the sets $X_n \subset E^d$ with the property $A_d(X_n) = A_d(n)$?

Problem 6. What are the exact values for $V_d(n)$, $d \geq 3$, $n \geq d+1$, and what is the description of the sets $V_n \subset E^d$ with the property $A_d(V_n) = V_d(n)$?

In 1962, Danzer and Grünbaum [9] introduced the notion of strictly antipodal points of a finite set in E^d and posed the question on the upper bound for $A'(d)$. Grünbaum [20] proved that $A'(3) = 5$. He also observed that $N'(d) \leq A'(d)$ and posed the following problem.

Problem 7. Is it true that $N'(d) = A'(d)$ for all $d \geq 2$?

Danzer and Grünbaum [9] gave an example of a set with $2d - 1$ points in E^d , any two of them being strictly antipodal. It was believed for a long time that $A'(d) = 2d - 1$, but Erdős and Füredi [14] showed that $A'(d) \geq \left\lceil \left(2/\sqrt{3}\right)^d / 2 \right\rceil$ (see Section 3).

Problem 8. What are the exact values for $A'(d)$, $d \geq 4$?

Regarding the value for $A'_d(n)$, Grünbaum [20] mentioned that $\lceil n/2 \rceil \leq A'_2(V_n) \leq n$. These inequalities were sharpened by Nguyễn and Soltan [32], who proved that

$A'_2(V_n) = n - k$, where k is the number of parallel sides of the polygon $\text{conv } V_n$. As a consequence, $A'_2(n) = n$.

For $d = 3$, Grünbaum [20] remarked that $A'_3(n) \geq \lfloor n/2 \rfloor \cdot \lfloor (n+1)/2 \rfloor + 2$ if $n \geq 4$. Makai and Martini [27] proved that

$$\lfloor n^2/3 \rfloor \leq A'_3(n) \leq 2n^2/5,$$

and for any $d \geq 4$,

$$\left(1 - \frac{\text{const}}{(1.0044)^d}\right) \frac{n^2}{2} - O(1) \leq A'_d(n) \leq \left(1 - \frac{1}{2^d - 1}\right) \frac{n^2}{2}.$$

They also conjectured that, for all $d \geq 2$,

$$A'_d(n) = \frac{n^2}{3} + O(1).$$

Problem 9. What are the exact values for $A'_d(n)$, $d \geq 3$, $n \geq d+1$, and what is the description of the sets $X_n \subset E^d$ with the property $A'_d(X_n) = A'_d(n)$?

If $V'_d(n)$ denotes the minimum of values $A'_d(V_n)$ taken over all convex d -polytopes with n vertices in E^d , then Nguyễn and Soltan [32] proved that $V'_d(n) \geq \lfloor n/2 \rfloor$, and that $V'_d(n) = \lfloor n/2 \rfloor$ holds for all even $n \geq 2d$ or for all odd $n \geq 4d - 1$ (see also [28]).

For small values of n , Nguyễn [31] proved the following relations:

$$V'_3(4) = V'_3(5) = V'_3(7) = V'_3(9) = 6 \text{ (cf. also [28])},$$

$$V'_d(d+1) = d(d+1)/2,$$

$$V'_d(n) \geq 3(d-1) \text{ if } d+2 \leq n \leq 2d-2,$$

$$V'_d(2d-1) = 2d,$$

$$V'_d(n) = 2d \text{ if } n \text{ is odd and } 2d+1 \leq n \leq 4d-1.$$

Problem 10. What are the exact values for $V'_d(n)$, $d \geq 3$, $n \geq d+1$, and what is the description of the sets $V_n \subset E^d$ with the property $A'_d(V_n) = V'_d(n)$?

References

- [1] N. Alon, J.H. Spencer, P. Erdős, The Probabilistic Method, Wiley, New York, 1992, MR 93h:60002.
- [2] K. Bezdek, T. Bisztriczky, K. Böröczky, Edge-antipodal 3-polytopes, in: J.E. Goodman, J. Pach, E. Welzl (Eds.), Discrete and Computational Geometry—Papers from MSRI Special Program, Cambridge University Press, Cambridge, 2004, pp. 1–6.
- [3] V. Boltyanski, I. Gohberg, Results and Problems in Combinatorial Geometry, Nauka, Moscow, 1965 (Russian). English translation: Cambridge University Press, Cambridge, 1985, MR 87c:52002.
- [4] V. Boltyanski, H. Martini, P. Soltan, Excursions into Combinatorial Geometry, Universitext, Springer, Berlin, 1997, MR 98b:52001.
- [5] T. Bonnesen, W. Fenchel, Theorie der konvexen Körper, Springer, Berlin, 1934.
- [6] C.D. Chakerian, H. Groemer, Convex bodies of constant width, in: P.M. Gruber, J.M. Wills (Eds.), Convexity and its Applications, Birkhäuser, Basel, 1983, pp. 49–96, MR 85f:52001.
- [7] H.T. Croft, On 6-point configurations in 3-space, J. London Math. Soc. 36 (1961) 289–306, MR 23#A4039.
- [8] B. Csikós, Edge-antipodal convex polytopes—a proof of Talata's conjecture, in: A. Bezdek (Ed.), Discrete Geometry, Marcel Dekker, New York, 2002, pp. 201–205.

- [9] L. Danzer, B. Grünbaum, Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V.L. Klee, *Math. Z.* 79 (1962) 95–99, MR 25#1488.
- [10] P. Erdős, On sets of distances of n points, *Amer. Math. Monthly* 53 (1946) 248–250, MR 7,471c.
- [11] P. Erdős, Problem 4306, *Amer. Math. Monthly* 55 (1948) 431.
- [12] P. Erdős, Some unsolved problems, *Michigan Math. J.* 4 (1957) 291–300, MR 20#5157.
- [13] P. Erdős, On sets of distances of n points in Euclidean space, *Magyar. Tud. Akad. Mat. Kutató Int. Közl.* 5 (1960) 165–169, MR 25#4420.
- [14] P. Erdős, Z. Füredi, The greatest angle among n points in the d -dimensional Euclidean space, *Ann. Discrete Math.* 17 (1983) 275–283, MR 87g:52018.
- [15] P. Erdős, J. Pach, Variations on the theme of repeated distances, *Combinatorica* 10 (1990) 261–269, MR 92b:52037.
- [16] P. Erdős, G. Purdy, Extremal problems in combinatorial geometry, in: R.L. Graham, M. Grötschel, L. Lovász (Eds.), *Handbook of Combinatorics*, vols. 1,2, Elsevier, Amsterdam, 1995, pp. 809–874, MR 96m:52025.
- [17] W. Fenchel, Lösung der Aufgabe Nr. 167, *Jber. Deutsch. Math.-Verein.* 45 (1935) 34–35.
- [18] B. Grünbaum, A proof of Vázsonyi's conjecture, *Bull. Research Council Israel, Sect. A* 6 (1956) 77–78.
- [19] B. Grünbaum, Borsuk's problem and related questions, in: V. Klee (Ed.), *Convexity*, Seattle, WA, 1961, *Proceedings of Symposia Pure Mathematics*, vol. VII, Amer. Math. Soc., Providence, RI, 1963, pp. 271–284, MR 27#4134.
- [20] B. Grünbaum, Strictly antipodal sets, *Israel J. Math.* 1 (1963) 5–10, MR 28#2480.
- [21] B. Grünbaum, *Convex Polytopes*, Interscience Publishers, New York, 1967, MR 37#2085.
- [22] H. Hadwiger, Ungelöste Probleme. Nr. 6, *Elem. Math.* 10 (1955) 88.
- [23] H. Hadwiger, Nachtrag zu Nr. 6, *Elem. Math.* 10 (1955) 111.
- [24] A. Heppes, Beweis einer Vermutung von A. Vázsonyi, *Acta. Math. Acad. Sci. Hungar.* 7 (1957) 463–466, MR 19,304e.
- [25] H. Hopf, E. Pannwitz, Aufgabe Nr. 167, *Jber. Deutsch. Math.-Verein.* 43 (1934) 114.
- [26] V. Klee, *Unsolved Problems in Intuitive Geometry*, Mimeographed Notes, Seattle, 1960.
- [27] E. Makai Jr., H. Martini, On the number of antipodal and strictly antipodal pairs of points in finite subsets of R^d , in: P. Gritzmann, B. Sturmfels (Eds.), *Applied Geometry and Discrete Mathematics. The Victor Klee Festschrift*, Amer. Math. Soc., Providence, RI, 1991, pp. 457–470, MR 92f:52020.
- [28] E. Makai Jr., H. Martini, On the number of antipodal and strictly antipodal pairs of points in finite subsets of R^d . II, *Period. Math. Hungar.* 27 (1993) 185–198, MR 95d:52016.
- [29] H. Martini, Shadow-boundaries of convex bodies, *Discrete Math.* 155 (1966) 161–172, MR 97j:52005.
- [30] C.C. Neaderhouser, G.B. Purdy, On finite sets in E^k in which the diameter is frequently achieved, *Period. Math. Hungar.* 13 (1982) 253–257, MR 84b:52015.
- [31] M.H. Nguyễn, *Affine Diameters of Convex Bodies*, (Russian) Ph.D. Thesis, State University of Moldova, Chişinău, 1990.
- [32] M.H. Nguyễn, V. Soltan, Lower bounds for the numbers of antipodal pairs and strictly antipodal pairs of vertices in a convex polytope, *Discrete Comput. Geom.* 11 (1994) 149–162, MR 84j:52017.
- [33] J. Pach, P.K. Agarwal, *Combinatorial Geometry*, Wiley, New York, 1995, MR 96j:52001.
- [34] K. Schütte, Minimale Durchmesser endlicher Punktmengen mit vorgeschriebenem Mindestabstand, *Math. Ann.* 150 (1963) 91–98, MR 26#5479.
- [35] S. Straszewicz, Sur un problème géométrique de P. Erdős, *Bull. Acad. Polon. Sci. Cl. III* 5 (1957) 39–40, MR 19,304f.
- [36] J.W. Sutherland, Lösung der Aufgabe Nr. 167, *Jber. Deutsch. Math.-Verein.* 45 (1935) 33–34.
- [37] I. Talata, On extensive subsets of convex bodies, *Period. Math. Hungar.* 38 (1999) 231–246, MR 2001b:52035.
- [38] A.C. Thompson, *Minkowski Geometry*, Cambridge University Press, Cambridge, 1996, MR 97f:52001.
- [39] D.R. Woodall, Trackles and deadlock, in: D. Welsh (Ed.), *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)*, Academic Press, London, 1971, pp. 335–347, MR 43#3154.
- [40] S.A. Yugaĭ, On the largest number of diameters of a point set in Euclidean space, *Investigations in Topological and Generalized Spaces*, Kirgiz. Gos. Univ., Frunze, 1988, pp. 84–87 (Russian), MR 1 165 340.